Notes on the Reflection and Refraction of SH Pulse emitted from a Point Source

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Abstract

The problem on the reflection and refraction at the interface between two elastic media of a SH pulse emitted from a point source is investigated. The general feature of the results seems to be essentially similar to that given by Pekeris.

C. L. PEKERIS (1941) investigated the propagation of a SH pulse due to a sudden application of a concentrated torsional stress, at the surface of a layered medium, and gave a figure illustrating the displacement at moderate distance from the source, in his brief abstract, without describing the process of calculation. In the present paper, we intend to deal with the reflection and refraction at the interface between two elastic media of the SH pulse, using the methods developed by T. SAKAI (1934) in his investigation of the propagation of the elastic waves in a semi-infinite elastic solid, and by T. SAKAI and S. SYÔNO (1935) in their study of the propagation of sound waves in the two layered atmosphere. The same method was used by us (1953) in the investigation of the reflection and refraction of the explosive sounds at the ocean bottom.

The interface is assumed to be horizontal, and the z-axis is directed vertically downward, and the ξ, η-axes and the x, y-axes are taken in the interface, where $\xi = x\cos\varphi + y\sin\varphi$. Let the density, rigidity, velocity of S wave, displacement and stress be denoted by ρ, μ, c, δ and T respectively. The suffices 1 and 2 refer to the upper medium and the lower one respectively.

The components of displacement and stress due to the plane SH wave propagating parallel to the ξ, z-plane are expressed by

$$\delta_\xi = 0, \qquad \delta_\eta = -\frac{\partial \boldsymbol{A}_z}{\partial \xi}, \qquad \delta_z = 0. \tag{1}$$

$$T_{z\xi} = 0, \qquad T_{z\eta} = -\mu\frac{\partial^2 \boldsymbol{A}_z}{\partial z \partial \xi}, \qquad T_{zz} = 0. \tag{2}$$

The boundary conditions to be satisfied at the interface $z=0$, are

$$\left.\begin{aligned} \left(-\frac{\partial \boldsymbol{A}_z}{\partial \xi}\right)_1 &= \left(-\frac{\partial \boldsymbol{A}_z}{\partial \xi}\right)_2, \\ \left(-\mu\frac{\partial^2 \boldsymbol{A}_z}{\partial z\partial \xi}\right)_1 &= \left(-\mu\frac{\partial^2 \boldsymbol{A}_z}{\partial z\partial \xi}\right)_2. \end{aligned}\right\} \tag{3}$$

Let us represent the incident plane SH wave by $\overset{(0)}{\boldsymbol{A}}_{z,1}$, that reflected back by $\overset{(1)}{\boldsymbol{A}}_{z,1}$ and that refracted into the lower medium by $A_{z,2}$, and put as following :

$$\left.\begin{aligned} A_{z,1}^{(0)} &= \exp\{-ik_1(\xi\sin w + z\cos w)\}\,, \\ A_{z,1}^{(1)} &= A\exp\{-ik_1(\xi\sin w_1 - z\cos w_1)\}\,, \\ A_{z,2} &= B\exp\{-ik_2(\xi\sin w_2 + z\cos w_2)\}\,, \end{aligned}\right\} \tag{4}$$

$$k_1 = \omega / c_1\,,\quad k_2 = \omega / c_2\,,\quad c_1 = \sqrt{\mu_1 / \rho_1}\,,\quad c_2 = \sqrt{\mu_2 / \rho_2}\,.$$

w is the incident angle, w_1 the reflected angle, and w_2 the refracted angle. The time factor $\exp(i\omega t)$ is here and often hereafter temporarily omitted.

Inserting (4) into (3), we have

$$k_1\sin w = k_1\sin w_1 = k_2\sin w_2\,,\quad w = w_1\,, \tag{5}$$

$$A = (m\gamma - \sqrt{n^2 - 1 + \gamma^2}) / D(\gamma)\,, \tag{6}$$

$$B = 2m\gamma / D(\gamma)\,, \tag{7}$$

$$D(\gamma) = \sqrt{n^2 - 1 + \gamma^2} + m\gamma\,, \tag{8}$$

$$\gamma = \cos w,\quad m = \mu_1 / \mu_2,\quad n = k_2 / k_1 = c_1 / c_2\,. \tag{9}$$

When we perform the operation

$$-\frac{ik_1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}+i\infty} dw \sin w \exp(-ik_1 d\cos w) \tag{10}$$

to $A_{z1}^{(0)}$ we get

$$A_{z1}^{(0)} = \exp(-ik_1 R) / R\,,$$

$$R = \sqrt{x^2 + y^2 + (z+d)^2} = \sqrt{r^2 + (z+d)^2}\,,\quad z + d \geqq 0\,. \tag{11}$$

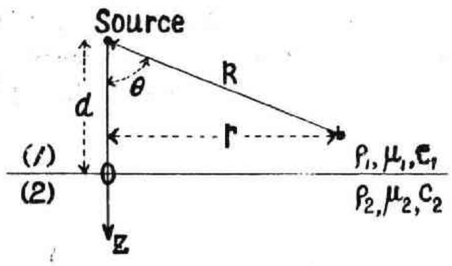

Fig 1.

The displacement δ which is horizontal and transverse to the direction of propagation corresponding to (11), is

$$\begin{aligned} \delta &= -\frac{\partial}{\partial r} A_{z1}^{(0)} \\ &= \sin\theta_0\,(ik_1/R + 1/R^2)\exp(-ik_1 R), \\ &\sin\theta_0 = r/R\,. \end{aligned} \tag{12}$$

(12) expresses a kind of the spherical SH wave emitted from the point source ($x=0$, $y=0$, $z=-d$).

Performing the same operation (10) to $A_{z,1}^{(1)}$ in (4) and putting $z=-d$, we have the expression of $A_{z,1}^{(1)}$ at (r, $z=-d$) of the SH wave which is reflected back from the interface, when the spherical SH wave (12) is incident on the surface. Thus,

$$A_{z1}^{(1)} = -\frac{ik_1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}+i\infty} A(\gamma)\exp\{-ik_1(\sin w\cos\varphi\cdot r + 2\cos w\cdot d)\}\sin w\, dw\,. \tag{13}$$

The horizontal and transverse component of displacement corresponding to (13), is

$$\delta = -\frac{\partial}{\partial \gamma} A_{z,1}^{(1)} . \tag{14}$$

(14) can be transformed into

$$\delta = -\frac{ik_1^2}{2}\int_{-\frac{\pi}{2}-i\infty}^{\frac{\pi}{2}+i\infty} \sin^2 w\, A(\gamma)\, H_1^{(1)} (k_1 r \sin w) \exp(-ik_1 2d \cos w)\, dw , \tag{15}$$

where $H_1^{(1)}$ is the HANKEL's function of the first kind and of the order 1.

(15) is similar to (27) of SAKAI and SYÔNO's paper. When $n<1$, and $k_1 r \gg 1$, $k_1 d \gg 1$, $\sin\theta > n$, the result of integration can be obtained as follows.

$$\delta = \delta_1 + \delta_2 , \tag{16}$$

$$\delta_1 = ik_1 \sin\theta \exp(i\alpha) \frac{\exp(-ik_1 R_1)}{R_1} , \tag{17}$$

$$\delta_2 = \frac{2n^2}{m(1-n^2)^{1/4}} \frac{\exp\{-i(k_2 r + 2k_1 d\sqrt{1-n^2})\}}{(\sin\theta)^{1/2}(\sqrt{1-n^2}\sin\theta - n\cos\theta)^{3/2}} \frac{1}{R_1^2} \tag{18}$$

$$\begin{gathered} R_1\cos\theta = 2d, \quad R_1 \sin\theta = r. \qquad R_1^2 = \sqrt{r^2+(2d)^2} , \\ \tan\alpha = 2m\cos\theta\sqrt{\sin^2\theta - n^2} / \{m^2\cos^2\theta - (\sin^2\theta - n^2)\} , \\ k_2 r + 2k_1 d\sqrt{1-n^2} = \omega\left(\frac{2l}{c_1} + \frac{r-2l\sin\theta_c}{c_2}\right), \\ l\cos\theta_c = d, \quad \sin\theta_c = n . \end{gathered} \tag{19}$$

Inserting the time factor $\exp(i\omega t)$ in (17) and (18), we see that δ_1 expresses the reflected SH wave, and δ_2 the so-called refracted wave. (Fig. 2).

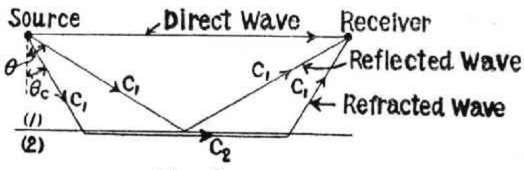

Fig 2.

In order to obtain the displacement D for the shock type wave which vary as $\psi(t) = F/(t^2+b^2)$ as to time t, we have to perform the operation

$$\frac{1}{\pi}\mathcal{R}\int_0^\infty d\omega \int_{-\infty}^{\infty} \psi(\sigma)\exp\{-i\omega(\sigma - t)\}\, d\sigma , \tag{20}$$

$$\psi(t) = \frac{F}{t^2+b^2}, \quad F>0, \quad b>0 . \tag{21}$$

to (12), (17) and (18). The results are as following :

$$D_0 = F\left\{\frac{1}{R^2 b^2(\tau_0^2+1)} - \frac{2}{c_1 R b^3}\frac{\tau_0}{(\tau_0^2+1)^2}\right\}, \quad \theta_0 = \frac{\pi}{2} , \tag{22}$$

$$D_1 = -\frac{F\sin\theta}{c_1 R_1 b^3}\left\{\frac{2\tau_1\cos\alpha + (1-\tau_1^2)\sin\alpha}{(\tau_1^2+1)^2}\right\}, \tag{23}$$

$$D_2 = \frac{2n^2 F}{m(1-n^2)^{1/4}} \frac{1}{(\sin\theta)^{1/2}(\sqrt{1-n^2}\sin\theta - n\cos\theta)^{3/2}} \frac{1}{R^2_1 b^2} \frac{1}{1+\tau_2^2} ,$$

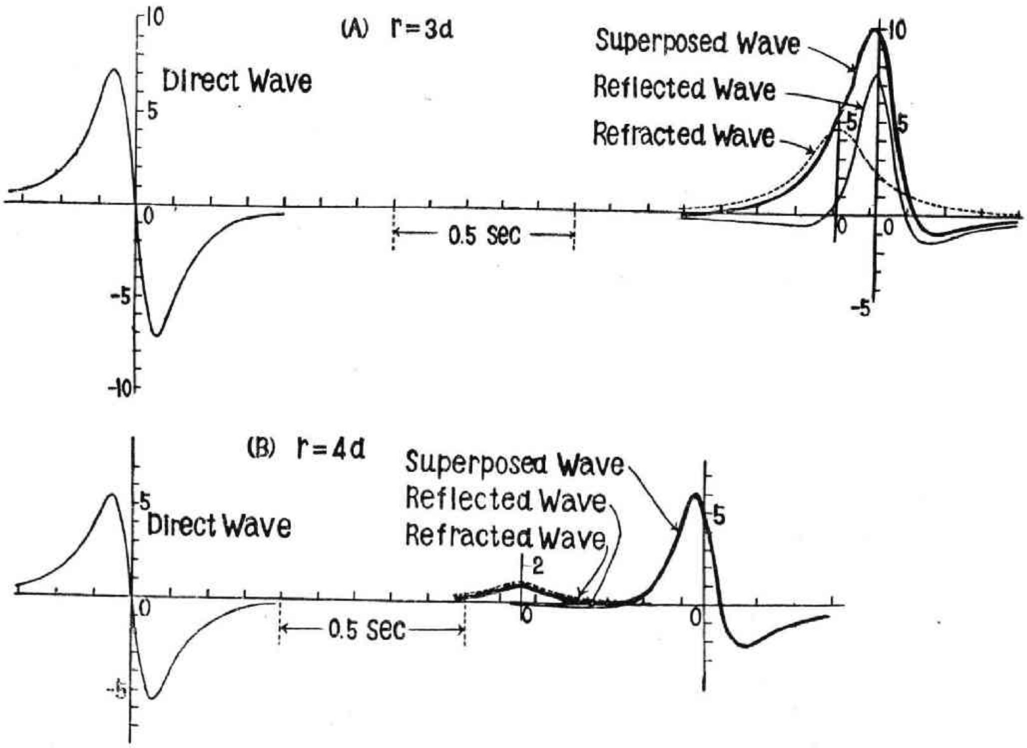

Fig 3.

$$\tau_0 = \left(t - \frac{R}{c_1}\right) \Big/ b, \ \tau_1 = \left(t - \frac{R_1}{c_1}\right) \Big/ b, \ \tau_2 = \left\{t - \left(\frac{2l}{c_1} + \frac{r - 2l \sin \theta_c}{c_2}\right)\right\} \Big/ b. \quad (24)$$

D_0, D_1 and D_2 expresses the direct, reflected and refracted SH pulse respectively.

Putting the values of some quantities as following,

$$v_1 = 3.0 \text{ km/sec.} \quad v_2 = 4.0 \text{ km/sec.} \quad \rho_1 = 2.7, \quad \rho_2 = 3.0,$$
$$d = 10 \text{ km}, \quad F = 1, \quad b = 0.1 \text{ sec.}$$

and hence

$$n = 0.75, \quad m = 0.506, \quad \theta_c = 48°35',$$

we calculated the displacement at the point $(r, z=-d)$ for the direct wave D_0, reflected wave D_1 and refracted wave D_2, for the cases $r=3d$ and $r=4d$. In the Fig. 3 are shown the results of numerical calculation. The curve in the Fig. 3 is essentially similar to that given by PEKERIS.

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